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E_{11} origin of brane charges and U-duality multiplets

Peter West

Department of Mathematics, King's College
London WC2R 2LS, U.K., and
Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9, 1090 Wien, Austria
E-mail: pwest@mth.kcl.ac.uk

ABSTRACT: We derive general equations which determine the decomposition of the G^{+++} multiplet of brane charges into the sub-algebras that arise when the non-linearly realised G^{+++} theory is dimensionally reduced on a torus. We apply this to calculate the low level E_8 multiplets of brane charges that arise when the E_8^{+++}, or E_{11}, non-linearly realised theory is dimensionally reduced to three dimensions on an eight dimensional torus. We find precise agreement with the U-duality multiplet of brane charges previously calculated, thus providing a natural eleven dimensional origin for the "mysterious" brane charges found that do not occur as central charges in the supersymmetry algebra. We also discuss the brane charges in nine dimensions and how they arise from the IIA and IIB theories.

KEYWORDS: M-Theory, String Duality.

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1. Introduction

Although there are a number of different approaches to string theory it has been clear for many years that none of them provides a complete formulation of string theory. For example, string field theory does, at least in principle, provide a non-perturbative description of string theory, but it does not readily accommodate non-trivial backgrounds. The IIA [1] and IIB [2, 3] supergravity theories on the other hand possess so much supersymmetry that they are essentially unique and as a result must contain all the low energy effects of the corresponding string theories. As such, the properties of these theories have provided much of our knowledge of what might constitute a proper formulation of string theory. The scalars in supergravity multiplets always belong to non-linear realisations and the groups that occur in these constructions were one of the most surprising developments in the construction of supergravity theories. The first time this was observed [4] was in the context of the $N = 4$ supergravity theory in four dimensions, however, perhaps the most celebrated example is the $N = 8$ supergravity theory in four dimensions whose scalars belong to the non-linear realisation of E_7 with respect to a A_7 sub-group [5]. One of the most important such examples for string theory is the $\mathrm{SL}(2, \mathbb{R})$ symmetry [2] of IIB supergravity theory and one of the most interesting from the view point of this paper concerns the scalars of the maximal supergravity theory in three dimensions which belong to a non-linear realisation of E_8 with respect to a D_8 sub-group [6].

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The eleven dimensional supergravity [7] dimensionally reduced on a circle leads to the IIA supergravity theory which possess a $\mathrm{SO}(1,1)$ coset symmetry, but on a k dimensional torus, for $k \leq 8$, it leads to a theory that possess E_d symmetry [8]. The IIB supergravity in ten dimensions is not related to eleven dimensional supergravity in such an obvious way. However, since there is only one maximally supergravity theory in nine dimensions, both the IIB supergravity with the IIA supergravity theories in ten dimensions must lead to this unique maximal theory when each is reduced on a circle. As such, one need not consider the dimensional reduction of the IIB theory separately.

The low energy effective action [9] for the heterotic string compactified to four dimensions possess an $\mathrm{SL}(2,\mathbb{R})$ symmetry. It was realised that this symmetry would be broken by quantum effects associated with solitons and it was conjectured that the full quantum theory would possess an $\mathrm{SL}(2,\mathbb{Z})$ symmetry. [10, 37]. Furthermore, it was realised that this symmetry contained what were called S-duality transformations that swopped perturbative with non-perturbative effects and visa versa [10, 37]. It was subsequently proposed [11] that the type II string theories possessed the corresponding $E_d,\ d = 1,\ldots,8$ symmetry found in the corresponding supergravity theories, but restricted to be over an appropriate integer valued field and that the IIB string theory possessed an $\mathrm{SL}(2,\mathbb{Z})$ symmetry. These symmetries became known as U-dualities and they include the well known T-duality symmetries [12] which are known to be a symmetry order by order of string perturbation theory [13] and thought also to hold for non-perturbative effects. The IIA string theory reduced on a $d-1$ dimensional torus is known to be invariant under a T duality transformation $\mathrm{SO}(d-1,d-1,\mathbb{Z})$. As such, M theory on a d torus should be invariant under $\mathrm{SO}(d-1,d-1,\mathbb{Z})$. However, it is also invariant under a $\mathrm{SL}(d,\mathbb{Z})$ symmetry which is the remnant of the general coordinate transformations preserved by the torus. Seen from the IIB theory, this latter symmetry contains the non-perturbative S-transformation of $\mathrm{SL}(2,\mathbb{Z})$. Hence, M theory reduced on a d torus should be invariant under the closure of $\mathrm{SO}(d-1,d-1,\mathbb{Z})$ with the $\mathrm{SL}(d,\mathbb{Z})$. The closure of these two groups generate the Weyl group of E_d which one can take to define the meaning of E_d taken over the appropriate field [15]–[18].

Carrying out such U-duality symmetry transformations on the solitons in the supergravity theories which correspond to the fundamental strings leads to the solutions for branes and, as a result, it become clear that string theory needed extending to a theory that contains branes as well as strings as fundamental objects. The branes in eleven dimensional supergravity and the IIA and IIB supergravity theories possess topological charges that occur in the supersymmetry algebra as central charges [14]. In the dimensional reduction of these theories on a d-dimensional torus the branes may wrap on part of, or sometimes all, of the torus and one then finds a more complicated set of branes in the dimensionally reduced theory. The charges of these branes are given in terms of the one parameter l_p of the eleven dimensional theory and the radii $R_i,\ d = 1,\ldots,8$ of the torus. Using the known formulae for the transformations of the radii and string coupling under the $\mathrm{SO}(d-1,d-1)$ T-dualities and $\mathrm{SL}(d,\mathbb{Z})$ transformations mentioned above allowed the authors of references [15]–[18] to compute the transformations of the brane charges. Indeed , it was by carry out this calculation that these authors found [15]–[18] that the

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closure of these two groups was isomorphic to the Weyl group of $E_d, d = 1, \ldots, 8$. Hence, by construction, they found a set of brane charges that belonged to representations of the Weyl group of $E_d, d = 1, \ldots, 8$.

In view of the result of reference [14] one might expect the brane charges to occur in the supersymmetry algebra. Indeed, for a dimensional reduction on T^d for $d \leq 7$ the content of the multiplet of point particle brane charges correspond precisely with the central charges that occurred in the supersymmetry algebra of the dimensionally reduced theory [15]–[19]. However, for $d \geq 8$ they found that the multiplet of brane charges contained more charges than there were central charges in the corresponding supersymmetry algebra. Furthermore, these additional charges did not correspond to charges of the familiar branes and so their meaning and origin was unclear [15]–[19]. For charges corresponding to strings and higher dimensional objects the mismatch occurs for dimensional reduction on a torus of even smaller dimension.

As explained in reference [19], these results are not in conflict with the construction of supergravity theories and their non-linear realised symmetries. The supercharges in these theories transform under the subgroup, i.e. SO(16) in the case of $d = 8$, associated with the non-linear realisation and as a result the central charges must, from the point of view of the supersymmetry algebra, only transform, and so form multiplets, under this subgroup. The fact that for dimensional reduction on a torus of sufficiently low dimension the central charges form multiplets of not just the subgroup, but the larger group that occurs in the non-linearly realised theory, i.e. the Weyl group of $E_d, d = 1, \ldots, 8$, suggests that the brane charges should in general form multiplets of the latter group. Indeed for U-duality to be true this would have to be the case. As a result, it is desirable to find a satisfactory origin for the "mysterious" charges that U-duality predicts.

Study of the properties of the eleven dimensional supergravity theory has lead to the conjecture [20] that M theory possess an E_{11} Kac-Moody symmetry. In particular, it was found that the bosonic sector of eleven dimensional supergravity theory could be formulated as a non-linear realisation [21]. The infinite dimensional algebra involved in this construction was the closure of a finite dimensional algebra, denoted G_{11}, with the eleven dimensional conformal algebra. The non-linear realisation was carried out by ensuring that the equations of motion were invariant under both finite dimensional algebras, taking into account that some of their generators were in common. The algebra G_{11} involved the space-time translations together with an algebra $\hat{G}_{11}$ which contained A_{10} and the Borel subalgebra of E_7 as subalgebras. The algebra $\hat{G}_{11}$ was not a Kac-Moody algebra, however, it was conjectured [4] that the theory could be extended so that the algebra $\hat{G}_{11}$ was promoted to a Kac-Moody algebra. It was shown that this Kac-Moody symmetry would have to contain a certain rank eleven Kac-Moody algebra denoted E_{11} [20].

Consequently, it was argued [20] that an extension of eleven dimensional supergravity should possess an E_{11} symmetry that was non-linearly realised. One advantage of this approach is that the symmetries found when the eleven dimensional supergravity theory was dimensionally reduced are already present in the eleven dimensional theory and so occur naturally . One of the advantages of a non-linear realisation is that the dynamics is largely specified by the algebra if the chosen local subalgebra is sufficiently large.

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The same analysis was applied to the IIA and IIB supergravity theories which were conjectured to be part of a larger theory which also possessed an E_{11} symmetry [20, 27]. This is consistent with the idea that the type II string theory in ten dimensions and an eleven dimensional theory are part of a single theory called M theory and indeed their common E_{11} origin provides explicit relations between the two theories [28].

Arguments similar to those advocated for eleven dimensional supergravity in [20] were proposed to apply to gravity [29] in D dimensions the effective action of the closed bosonic string [4] generalised to D dimensions and the heterotic string [38] and the underlying Kac-Moody algebras were identified. It was realised that the algebras that arose in all these theories were of a special kind and were called very extended Kac-Moody algebras [30]. Indeed, for any finite dimensional semi-simple Lie algebra $\mathcal{G}$ one can systematically extend its Dynkin diagram by adding three more nodes to obtain an indefinite Kac-Moody algebra denoted $\mathcal{G}^{+++}$. In this notation E_{11} is written as E_8^{+++}. The algebras for gravity and the closed bosonic string being A_{D-3}^{+++} [29] and D_{D-2}^{+++} [20] respectively.

It was proposed in [20, 29, 30, 38] and [26, 31, 32, 25], that the non-linear realisation of any very extended algebra $\mathcal{G}^{+++}$ leads to a theory, called $\mathcal{V}_{\mathcal{G}}$ in [32], that at low levels includes gravity and the other fields and it was hoped that this non-linear realisation contains an infinite number of propagating fields that ensures its consistency. Indeed, it was shown [32] that the low level content of the adjoint representation of $\mathcal{G}^{+++}$ predicted a field content for a non-linear realisation of $\mathcal{G}^{+++}$ which was in agreement with the oxidized theory associated with algebra $\mathcal{G}$.

Some papers have uncovered relationships between the solutions in the oxidised theories and the $\mathcal{G}^{+++}$ symmetry conjectured to be present in their extension. In reference [26], the non-linear realisation of $\mathcal{G}^{+++}$ restricted to its cartan subalgebra was constructed and the resulting Weyl transformations were shown to transform the moduli of the Kasner solutions into each other. Furthermore, for $E_8^{+++} = E_{11}$ and D_{24}^{+++} these Weyl transformations were shown to be the U-duality transformations in the corresponding string theories.

The question of how space-time was to be encoded in the theory was taken up in reference [22] where it was proposed that one should take the non-linear realisation of semi-direct product of E_{11} and a set of generators transforming in the l_1 representation of E_{11} where l_1 is the fundamental representation of E_{11} corresponding to the very extended node. The lowest level generator in this representation is the space-time translation operator, at the next two levels it contains the two central charges that occur in the eleven dimensional supersymmetry algebra and it also contains an infinite number of higher level object. Hence the first three objects in the l_1 representation can be interpreted as the charges associated with the point particle, two brane and five brane of the eleven dimensional theory. It was shown in reference [23] that all the objects in the l_1 representation have the correct A_{10} structure to be interpreted as charges for all the branes whose sources are fields in the non-linearly realised theory. An alternative idea for encoding space was to regard it to arise from the dynamics of E_{10} [24] or all of space-time to arise from the dynamics of E_{11} [25]. A discussion of the different approaches can be found in reference [23].

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In this paper we use the approach of reference [22] and take the l_1 representation of E_{11} to contain the brane charges. In section 2 we derive formulae that can be used to find the low level content of the l_1, and the adjoint representations of G^{+++}, in terms of decompositions of G^{+++} that are appropriate for the dimensional reduction of the non-linear realised theory on tori. In section 3, we calculate the low level content of the l_1 representation of E_{11} in terms of its $E_8 \otimes A_2$ sub-algebra which is the one appropriate to the torus dimensional reduction of the non-linearly realised theory to three dimensional space-time. As such, we are find at low levels the brane charges of the non-linearly realised theory. We find in addition to the expected brane charges which are central charges in the supersymmetry algebra all the "mysterious charges" found in references [15]–[18] on the basis of U-duality considerations. Thus we find that the non-linearly realised E_{11} theory and its l_1 representation provide a natural explanation for the additional brane charges that are required if U-duality is to hold in M theory.

In section 4, we carry out a similar calculation for the decomposition appropriate to torus dimensional reduction to nine space-time dimensions and calculate the low level brane charges predicted by the E_8^{+++} non-linearly theory. We also derive the brane charges from a IIB perspective and derive relations between these and the charges in the IIA theory that are in agreement at low levels with that found in references [30, 40, 19]. We also find that a truncation of the E_8^{+++} non-linearly theory in nine dimensions which makes contact with the BPS extended theory considered in references [40, 19].

2. Decompositions of the l_1 representation of G^{+++}

As explained in the introduction, the objects that occur in the l_1 representation, that is the fundamental representation associated with the very extended node, are the brane charges for the non-linear realised theory based on G^{+++}. In this section, we derive general equations for calculating the low level content of the l_1 representation when decomposed into representations of the sub-algebras of G^{+++} that arise when the theory is dimensional reduced on a tori. We will study in detail the most useful such decomposition into $G \otimes A_2$. This is one of the most instructive decompositions as it contains the largest finite dimensional semi-simple Lie algebra contained in G^{+++} and so it allows us to see the most complete structure of this representation when expressed in terms of a Lie algebra that is well understood. We can think of this as a decomposition of the theory in its original dimension, but it corresponds to the decomposition that occurs when the theory is reduced to three dimensional space-time on a eight torus. In this case, the internal symmetry is G and the A_2 part is related to the representations of the Lorentz algebra in three dimensional space-time.

2.1 General decomposition

In reference [30] is was proposed to study lorentzian Kac-Moody algebras which are algebras whose Dynkin diagrams C possess at least *one* preferred node whose deletion leaves a, possible disconnected, Dynkin diagram C_R which is made up of the Dynkin diagrams all of which are those of finite dimensional semi-simple Lie algebras with possible exception of at most one affine Lie algebra. The properties of the lorentzian Kac-Moody algebra were then studied in terms of the, possibly reducible, algebra corresponding to the Dynkin diagrams that remain after the deletion of the preferred point. In this spirit, the notion of a level was introduced in the context of E_{10} in reference [24] and for any Kac-Moody algebra in [33]. The level of a root of the Kac-Moody algebra is just and number of times the simple root of the preferred node enters. The level of a generator being that of its associated root. The algebra is then studied level by level in terms of the remaining algebra. The low level generators of E_{10} and E_{11} have been studied in terms of A_9 [24, 34] and A_{10} [33, 34] respectively by deleting the exceptional node and for all very extended algebras G^{+++} in [32]. So far these level decompositions have been studied when the node deleted leaves just one semi-simple finite dimensional Lie algebra, However, for the application we wish to consider in this paper, we will delete a node such that the remaining Dynkin diagram C_R is not irreducible but contains the Dynkin diagrams of two or more semi-simple finite dimensional algebras $G^{(p)}$. The roots, weights and some other quantities was calculated in terms of the remaining algebra in reference [30] for this case, but the extension to give the corresponding level decomposition was not given.

To find the decomposition of the l_1 fundamental representation of G^{+++}, it is advantageous to consider the adjoint representation of G^{++++} [23] from which the former may be extracted in a simple way as explained below. The latter has a Dynkin diagram that is found from the Dynkin diagram C of G^{+++} by adding a new node attached to the very extended node by a single line. We label the additional node by $*$.

The non-linear realisation of G^{+++} contains gravity which is associated with a preferred A_{D-1} sub-algebra where D is the space-time dimension in which the resulting theory lives. The Dynkin diagram of this A_{D-1} sub-algebra contains the very extended node of the Dynkin diagram of G^{+++} as well as $D-2$ other nodes attached to the very extended node and to each other by a single line. This line of connected dots has become known as the gravity line. For a given G^{+++}, there is in general more than one possible choice for this sub-algebra, or gravity line.

In this section we wish to consider the decomposition of G^{+++} and its l_1 representation in terms of the sub-algebra that results from the decomposition of the kth dot along the gravity line counting from the very extended node. This is the one appropriate to the decomposition of the non-linear realisation to k space-time dimensions since the remnant of the gravity line which includes the very extended node has $k-1$ dots and so corresponds to A_{k-1}, or $\mathrm{SL}(k)$. In general, the Dynkin diagram C_R that results from deleting this node will contain several pieces which we label by $p, q = 1, 2, \ldots$, their Dynkin diagrams being labeled by $C^{(p)}$ and the corresponding sub-algebras being $G^{(p)}$. The nodes of $C^{(p)}$ are labeled by the indices $i, j, \ldots = 1, 2, \ldots$. The index range will be different for each $C^{(p)}$ and the ambiguities of labeling of the indices on a given object are resolved by the knowledge of Dynkin diagram to which the object is associated.

We label the simple roots and weights of $G^{(p)}$ by $\alpha_i^{(p)}$ and $\lambda_i^{(p)}$, the range of the index i being apparent from the object to which it is attached. The Dynkin diagram C of G^{+++} is then labeled in a way which is appropriate to the decomposition we wish to perform; we label the node which is to be deleted by c while the remaining nodes, which belong to the sub-diagrams $C^{(p)}$, are labeled as for the each sub-diagram. In the cases of most interest to us the remaining Dynkin diagram C_R will contain two pieces. The very extended node will be contained in the $C^{(1)}$ diagram.

As explained above, we introduce an extra node, labeled by $*$, attached to the very extended node by a single line and consider the algebra G^{++++} so that we may consider the l_1 representation of G^{+++}. The roots of G^{++++} may be written as

$$\beta = m_* \alpha_* + \sum_a n_a \alpha_a \,, \tag{2.1}$$

where α_* is the simple root corresponding to the additional node and α_a are the roots of G^{+++}. For a positive root m_* and n_a are positive integers.

The roots of G^{++++} with level $m_* = 0$ are just roots of G^{+++}. Clearly, the commutator of a level $m_* = 1$ generator with level $m_* = 0$ generator gives a generator of level $m_* = 1$, as such the $m_* = 1$ generators form a representation of G^{+++}. It is the fundamental representation with highest weight $\lambda_1^{G^{+++}}$ of G^{+++} associated with the very extended node of G^{+++}, i.e. the l_1 representation. We can think of m_* as a level, but we will only be interested in the case $m_* = 1$ or $m_* = 0$.

In order to carry out the decomposition from G^{++++} to G^{+++} we write

$$\alpha_* = y - \lambda_1^{G^{+++}} \,, \tag{2.2}$$

where y is a vector that is orthogonal to the roots of G^{+++}. Since $\alpha_*.\alpha_* = 2$ and $\lambda_1^{G^{+++}}.\lambda_1^{G^{+++}} = \frac{1}{2}$, we conclude that $y^2 = \frac{3}{2}$.

The decomposition found by deleting the kth node of the gravity line which we referred to as node c, with simple roots α_c, proceeds as explained in reference [30]. We may write the simple roots of G^{+++} as

$$\alpha_c = x - \nu, \ \alpha_i^{(p)} \,, \tag{2.3}$$

where $\alpha_i^{(p)}$ are the roots of the algebras $G^{(p)}$. In the above

$$\nu = -\sum_{i,p} A_{ci(p)}^{G^{+++}} \lambda_i^{(p)} \frac{(\alpha_c, \alpha_c)}{(\alpha_i^{(p)}, \alpha_i^{(p)})} \,, \tag{2.4}$$

where $A_{ci(p)}^{G^{+++}}$ is the cartan matrix of G^{+++} labeled as explained above. This formula differs from that of reference [30] in that is includes the possibility of the algebra G being non-simply laced. We note that the simple roots of G^{+++} for nodes other than c are just the simple roots of the sub-algebras $G^{(p)}$, x is orthogonal to all of these and has its length determined by the requirement that the length of α_c be as required. The fundamental weights of G^{+++} are then given in terms of x and $\lambda_i^{(p)}$ by

$$l_i^{(p)} = \lambda_i^{(p)} + \frac{\nu \cdot \lambda_i^{(p)}}{x^2} x \,, \qquad l_c = \frac{x}{x^2} \,. \tag{2.5}$$

We note that $l_1^{(1)} = \lambda_1^{G^{+++}}$.

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Using the above expressions, we may write any root β of G^{+++} as given in equation (2.1) in the form

$$\beta = m_* y + x\left(m_c - m_* \frac{\nu^{(1)} \cdot \lambda_1^{(1)}}{x^2}\right) - \sum_p \Lambda^{(p)} \tag{2.6}$$

where

$$\Lambda^{(p)} = -\sum_i m_i^{(p)} \alpha_i^{(p)} + m_c \nu^{(p)} + \delta_{(1,p)} m_* \lambda_1^{(1)}\,, \tag{2.7}$$

where $\nu^{(p)}$ is the component of ν in the sub-algebra $G^{(p)}$, i.e. $\nu^{(p)} = -\sum_i A_{ci(p)}^{G^{+++}} \lambda_i^{(p)}$.

We are decomposing the representation of G^{++++} in terms of $G^1 \otimes G^2 \otimes \cdots$ and if a representation of $G^{(p)}$ occurs in the decomposition of the l_1, or adjoint, representations of G^{+++}, then its highest weight must occur in $\Lambda^{(p)}$ in both the positive and negative root spaces of G^{+++}. Applying this to the negative roots, we must find that there exists a root of G^{+++} such that

$$\Lambda^{(p)} = \sum_i p_i^{(p)} \lambda_i^{(p)}\,, \tag{2.8}$$

where $p_i^{(p)}$ are positive integers. Taking the scalar product with the fundamental weights of $G^{(p)}$ we then find the conditions

$$\sum_i p_i^{(p)} \lambda_i^{(p)} \cdot \lambda_j^{(p)} - m_c \nu^{(p)} \cdot \lambda_j^{(p)} - \delta_{(1,p)} m_* \lambda_1^{(1)} \cdot \lambda_j^{(1)} = -m_j^{(p)} \frac{2}{\left(\alpha_i^{(p)}, \alpha_i^{(p)}\right)}, \tag{2.9}$$

where $p_i^{(p)}$, $m_i^{(p)}$, m_* and m_c are all positive integers. The scalar products of the fundamental weights are related to the inverse cartan matrices of the sub-algebras by

$$\left(A^{(p)}\right)^{-1}_{ij} = \frac{2}{\left(\alpha_i^{(p)}, \alpha_i^{(p)}\right)} \left(\lambda_i^{(p)}, \lambda_j^{(p)}\right), \qquad \text{and} \qquad \left(\alpha_i^{(p)}, \lambda_j^{(q)}\right) = \frac{2\delta^{pq}\delta_{ij}}{\left(\alpha_i^{(p)}, \alpha_i^{(p)}\right)} \tag{2.10}$$

For finite dimensional semi-simple algebras, the inverse cartan matrices are positive definite and as a result the above equation tightly constrains the possible Dynkin indices $p_i^{(p)}$, or representations of the sub-algebras $G^{(p)}$ that can arise.

The above decomposition does not apply when one deletes the second node along the gravity line as the fundamental weight associated with this node has length zero and so can not be written as $\frac{x}{x^2}$.

The length squared of the roots of G^{++++} which contain the highest weights of the sub-algebras are given by

$$\beta^2 = \frac{3}{2} m_*^2 + x^2 \left(m_c - m_* \frac{\nu^{(1)} \cdot \lambda_1^{(1)}}{x^2}\right)^2 + \sum_q \sum_{i,j} p_i^{(q)} \lambda_j^{(q)} \cdot \lambda_j^{(q)} p_j^{(q)}\,. \tag{2.11}$$

This quantity is an integer and is bounded from above for all Kac-Moody algebras and for simply laced Kac-Moody algebras it can only take the values $2, 0, -2, \ldots$. Hence, we find a further constraint on the possible Dynkin labels $p_j^{(q)}$ and so representations that can arise.

A hyperbolic Kac-Moody algebras possess roots at all these values of β^2, but this is not so for more general Kac-Moody algebras. For a general Kac-Moody algebra the roots are either real or imaginary. By definition a real root not only has $\beta^2 = 2$, but must also be conjugate under the Weyl group to a simple root. An imaginary root has $\beta^2 \le 0$, but must also be conjugate under the Weyl group to a root that is in the fundamental Weyl chamber and also has connected support on the Dynkin diagram of the Kac-Moody algebra. We consider the integer coefficients of a root when expressed in terms of simple roots. For those integer coefficients that are zero we delete the corresponding nodes of the Dynkin diagram. A connected root is one for which the resulting is a connected diagram [36]. We will see that not every solution we find will respect this condition so we will discard such solutions.

Although equations (2.9) and (2.11) are necessary conditions for a representation to arise they are not sufficient. Indeed, in the case of the adjoint representation of G^{+++} they do not encode all the consequences of the Serre relations. However, we know from experience that the solutions to the above equations usually arise in the actual decomposition. We note that the above formalism does not predict the number of times a representation can arise at a given level.

The most common case, and the one of most interest to us, is when the deletion of the node on the gravity line leads to only two sub-algebras. It is useful in this case to introduce a more easily understood notation. We denote the roots, weights and Dynkin indices of the sub-algebras by

$$\alpha_i^{(1)} = \beta_i\,,\ \lambda_i^{(1)} = \mu_i\,,\ p_i^{(1)} = q_i\,,\ m_i^{(1)} = m_i\,;\ \ \alpha_i^{(2)} = \alpha_i\,,\ \lambda_i^{(2)} = \lambda_i\,,\ p_i^{(2)} = p_i\,,\ m_i^{(2)} = n_i\,. \tag{2.12}$$

We will also restrict our attention from now on to when G is simply laced and so all the simple roots of G^{+++} have length squared two. In this notation equation (2.9) becomes

$$\sum_i q_i \mu_i \cdot \mu_j - m_c \nu^{(1)} \cdot \mu_j - m_* \mu_1 \cdot \mu_j = -m_j \tag{2.13}$$

for the first sub-algebra and

$$\sum_i p_i \lambda_i \cdot \lambda_j - m_c \nu^{(2)} \cdot \lambda_j = -n_j \tag{2.14}$$

for the second sub-algebra.

2.2 Decomposition into $G \otimes A_2$

In this section, we derive the equations that allow the calculation of the content of the l_1 representation decomposed into representations of $G \otimes A_2$. This is the decomposition into the largest finite dimensional semi-simple Lie algebra contained in G^{+++} and so it allows us to see the most complete structure of this representation when expressed in terms of a Lie algebra that is well understood. We can think of this as a decomposition of the theory in its original dimension, but it corresponds to the decomposition that occurs when the theory is dimensionally reduced to three dimensional space-time. In this case, the internal symmetry is G and the A_2 part is related to the representations of Lorentz algebra in three space-time dimensions. As explained in appendix B, the Lorentz group arises essentially as the cartan involution invariant sub-algebra of A_2. The A_2 representations lead to representations of the Lorentz algebra that transform under the Lorentz algebra as the A_2 indices might naively suggest.

This decomposition results from the deletion of the third node in the gravity line. The gravity line begins at the very extended node of G^{+++} and then contains the over extended node and then the affine node which is the one to be deleted. Since by construction no other nodes are attached to the very extended and over extended nodes in the Dynkin diagram of G^{+++} this deletion leads to the sub-algebra $G \otimes A_2$ as claimed. As explained above, we label the affine node by c, the very extended and over extended nodes of G^{+++} which make up the algebra A_2 by $1, 2$ respectively and label the nodes of G^{+++} which form G after the deletion by $1, 2, \dots$.

Following equation (2.3), we take

$$\alpha_c = x - \nu \,, \quad \text{where} \quad \nu = \nu^{(1)} + \nu^{(2)} \,, \qquad \text{and} \qquad \nu^{(1)} = \mu_2 \,, \quad \nu^{(2)} = \theta = \sum_j c_j \lambda_j \,. \tag{2.15}$$

In this equation θ is the highest root of G and we have used the relation that $A^G_{cj} = -(\theta, \alpha_j)$ as c is the affine node of G. For the simply laced algebras we are studying here $\theta^2 = 2$. Since $\alpha_c^2 = 2$ we find that $x^2 = -2/3$.

Equation (2.14) can be written as

$$\sum_i p_i((A^G)^{-1})_{ij} - m_c \sum_k c_k((A^G)^{-1})_{kj} = -n_j \tag{2.16}$$

for $p_i, n_j = 0, 1, 2, \dots$ for fixed value of the level $, m_c = 0, 1, 2, \dots$. While equation (2.13) becomes

$$\sum_{i=1}^{2} q_i((A^{A_2})^{-1})_{ij} - m_c((A^{A_2})^{-1})_{2j} - m_*((A^{A_2})^{-1})_{1j} = -m_j \tag{2.17}$$

for $q_i, m_j = 0, 1, 2, \dots$ for fixed value of the level $m_*, m_c = 0, 1, 2, \dots$. As explained above, since the inverse cartan matrix of any finite dimensional semi-simple Lie algebra is positive definite equation (2.16) allows only a finite number of solutions for a given m_c. We note that it does not depend on m_*. As such, we may calculate, at low levels, the possible G representations present using this equation.

Explicitly writing out equation (2.17) we find that it becomes

$$2q_1 + q_2 - 2m_* - m_c = -3m_1 \,, \qquad q_1 + 2q_2 - m_* - 2m_c = -3m_2 \,. \tag{2.18}$$

As the m_i and q_i are positive, and as previously observed, there clearly are only a finite number of solutions for fixed m_c and m_*.

Furthermore, as β is a root of G^{++++} it must have length squared $2, 0, -2, \dots$ and so equation (2.1.11) becomes

$$\beta^2 = a + \sum_{i,j} p_i((A^G)^{-1})_{ij} p_j + \sum_{i,j=1}^{2} q_i((A^{A_2})^{-1})_{ij} q_j = 2, 0, -2, \dots \tag{2.19}$$

where $a = \frac{3}{2} m_*^2 - \frac{1}{6}(m_* + 2m_c)^2$.

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m_c	(q_1,q_2)	(m_1,m_2)	$\sum qA^{-1}q$
0	$(1,0)$	$(0,0)$	$\frac{2}{3}$
1	$(1,1)$	$(0,0)$	2
1	$(0,0)$	$(1,1)$	0
2	$(0,1)$	$(1,1)$	$\frac{2}{3}$
2	$(1,2)$	$(0,0)$	$4+\frac{2}{3}$
2	$(2,0)$	$(0,1)$	$2+\frac{2}{3}$
3	$(1,0)$	$(1,2)$	$\frac{2}{3}$
3	$(0,2)$	$(1,1)$	$2+\frac{2}{3}$
3	$(2,1)$	$(0,1)$	$4+\frac{2}{3}$
3	$(1,3)$	$(0,0)$	$8+\frac{2}{3}$
4	$(0,0)$	$(2,3)$	0
4	$(0,3)$	$(1,1)$	6
4	$(1,1)$	$(2,1)$	2
4	$(1,4)$	$(0,0)$	$6+\frac{2}{3}$
4	$(2,2)$	$(1,0)$	8

Table 1: Solutions of the A_2 equation for $m_* = 1$.

m_c	(q_1,q_2)	(m_1,m_2)	$\sum qA^{-1}q$
2	$(0,2)$	$(0,0)$	$2+\frac{2}{3}$
2	$(1,0)$	$(0,1)$	$\frac{2}{3}$
3	$(0,3)$	$(0,0)$	6
3	$(1,1)$	$(0,1)$	2
3	$(0,0)$	$(1,0)$	0

Table 2: Solutions of the A_2 equation for $m_* = 0$.

We are interested in the decomposition of the l_1 representation of G^{+++} for which we take $m_* = 1$. For $m_* = 1$ and $m_c = 1,2,3,4$ the solutions to equation (2.17), or equivalently (2.18), are given in table 2.1

We also give the solutions at low levels to A_2 equation (2.2.4) for $m_* = 0$ which corresponds to the adjoint representation of G^{+++}. There are in fact no solutions for $m_c = 0,1$, the higher level solutions are given in the table 2.

Equation (2.16) is much more complicated to solve. However, the results only depend on the level m_c, the inverse cartan matrix of G and the highest root θ, or equivalently the coefficients c_k, of G. For simply laced algebras, the highest root $\theta = \nu^{(2)}$ can be expressed in terms of the fundamental weights as follows

G	E_8	E_7	E_6	A_n	D_n	D_3
θ	λ_1	λ_6	λ_6	$\lambda_1+\lambda_n$	λ_2	$\lambda_2+\lambda_3$

The only solution for level $m_c = 0$ is $p_i = 0$, while for $m_c = 1$ there is the obvious solution namely $p_k = c_k$. For both of these solutions $n_k = 0$. More solutions for the case of E_8 are discussed at the end of this section.

Having solved equations (2.16) and (2.18) we can substitute the results for q_i and p_i into the remaining constraint of equation (2.19) and see which of these solutions are allowed. The values of the first term a in this equation for the l_1 representation, i.e. $m_* = 1$, are listed below for low values of m_c:

m_c	0	1	2	3	4
a	$1+\frac{1}{3}$	0	$-2-\frac{2}{3}$	$-6-\frac{2}{3}$	-12

Taking $m_c = 0$, we find that equation (2.18) can only be solved for $(q_1 = 1, q_2 = 0)$, while the only solution to equation (2.16) is $p_i = 0$. We find that these do indeed solve equation (2.19) for a G^{++++} root which is given by $\beta = (1, 0, \ldots, 0)$ which has $\beta^2 = 2$. This corresponds to an element which is a one rank tensor of A_2 and so a vector of the space-time group $\mathrm{SO}(1,2)$, but is inert under G. This root is the highest weight A_{10} state of G^{++++} corresponding to P_1. As a result, the solution we have found can be identified with $P_a, a = 1, 2, 3$.

Taking $m_c = 1$, we have the solution $p_k = c_k$ for equation (2.16) for which $\Lambda = \theta$ and so $\sum pA^{-1}p = 2$. As a result, only one of the two solutions to equation (2.18), listed in table 1, is allowed in equation (2.19) namely for the values $(q_1 = 0, q_2 = 0)$. The corresponding G^{+++} root is given by $\beta = (1, 1, 1, 1, c_1, c_2, \ldots)$ which has $\beta^2 = 2$. This is a scalar under $\mathrm{SO}(1,2)$, but has highest weight θ of G and so belongs to the adjoint representation of G.

m_c	p_i	$\sum pA^{-1}p$
0	$p_1 = 0$	0
1	$p_1 = 1$	2
1	$p_i = 0$	0
2	$p_1 = 1$	2
2	$p_1 = 0$	0
2	$p_7 = 1$	4
3	$p_1 = 1$	2
3	$p_7 = 1$	4
3	$p_8 = 1$	8
3	$p_2 = 1$	6
3	$p_i = 0$	0
4	$p_2 = 2$	24
4	$p_3 = 1$	12
4	$p_4 = 1$	20
4	$p_6 = 1$	14
4	$p_7 = 1$	4
4	$p_7 = 2$	16
4	$p_8 = 1$	8
4	$p_2 = 1 = p_7$	18
4	$p_2 = 1$	6
4	$p_i = 0$	0
4	$p_1 = 1 = p_2$	14
4	$p_1 = 1 = p_3$	22
4	$p_1 = 1 = p_7$	10
4	$p_1 = 1 = p_8$	16
4	$p_1 = 1$	2

Table 3: Solutions of the E_8 equation.

For future use we give in the table 3 the solutions to equation (2.18) for the case of E_8 up to level four.

We also have the solutions $p_1 = m_c - r$, all other p_i's are zero, with $\sum pA^{-1}p = 2(m_c - r)^2$, whenever p_1 is positive and for integer r. For $p_1 - m_c = 1$ these are the only other solutions, but for $p_1 - m_c = 2$ we can also have the solution with $p_2 = 1$, for which $\sum pA^{-1}p = 2(m_c - 2)^2 + 6(m_c - 2) + 6$, and the solution $p_7 = 1$ for which $\sum pA^{-1}p = 2(m_c - 2)^2 + 4(m_c - 2) + 4$. In fact, we have included some of these solutions above where it was useful to do so.

3. Brane charges in three dimensions

It has been known for many years that eleven dimensional supergravity dimensionally reduced on a eight-dimensional torus leads to a theory in three space-time dimensions that possess an E_8 symmetry [6]. It has been conjectured that there should exist an extension

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of the maximal supergravity theory in three space-time dimensions that includes string as well as branes that is invariant under an E_8 symmetry defined over an appropriate integer field [11], called U-duality.

It has been proposed [15]–[18] that the U-duality transformations should be generated by the $\mathrm{SL}(8,\mathbb{Z})$ remnant of general coordinate transformations preserved by the eight torus with radii $R_a, a = 1, \ldots 8$;

$$R_a \leftrightarrow R_{a+1}\,, \quad a = 1, \ldots, 7\,, \tag{3.1}$$

together with the double T-duality transformations of the type IIA theory found after the reduction on the first circle

$$R_a \to \frac{l_s^2}{R_a}\,, \quad R_b \to \frac{l_s^2}{R_b}\,, \quad g_s \to \frac{g_s l_s^2}{R_a R_b}\,, \quad a, b = 2, \ldots, 8 \tag{3.2}$$

all other radii unchanged. Here l_s is the string scale and g_s is the string coupling constant. Using the relations $l_p^3 = g_s l_s^3$ and $R_1 = g_s l_s$ which relate the eleven dimensional Planck length l_p and the radius R_1 of the circle used to reduce to the IIA theory we can relate the IIA variables to those of eleven dimensions to rewrite equation (3.2) as

$$R_a \to \frac{l_p^3}{R_b R_c}\,, \quad R_b \to \frac{l_p^3}{R_c R_a}\,, \quad R_c \to \frac{l_p^3}{R_a R_b}\,, \quad l_p^3 \to \frac{l_p^6}{R_a R_b R_c}\,, \qquad a, b, c = 1, \ldots, 8 \tag{3.3}$$

all other radii being unchanged. In deriving this last equation we have used the possibility to swop radii using equation (3.1). It has been shown [15]–[18] that the closure of the transformations of equations (3.1) and (3.3) is a group that is isomorphic to the Weyl transformations of E_8 and observed that in this approach one defines a symmetry which is automatically over an integer field.

The dimensional reduction of the point particle, two and fives branes of the eleven dimensional theory lead to a more rich structure of branes in three dimensions as these branes may wrap around different directions of the eight-dimensional torus in the dimensional reduction procedure. The charges of the branes in the three dimensional theory should belong to multiplets of the Weyl group of E_8, as well as belong to representations of $\mathrm{SO}(1,2)$ Lorentz symmetry of space-time, if the conjectures on U-duality are to be true. Starting with the known charges that do arise from dimensional reduction of the branes of the eleven dimensional supergravity theory, the authors of reference [15]–[18] used the U-duality transformations of equation (3.1) and (3.2) to find the complete E_8 Weyl group charge multiplets for the point particles and strings of the three dimensional theory. They found that the point particle charges belong to the adjoint, or λ_1, representation of E_8 while the string charges belong to the 3875, or λ_7 representation, of E_8. However, the authors of reference [15]–[18] found that these E_8 Weyl multiplets of brane charges contained more brane charges that one would expect from the dimensional reduction of the branes of the eleven dimensional supergravity theory. Put another way the brane charges that arise from the dimensional reduction did not form multiplets of E_8, and so some of the brane charges in the E_8 Weyl group multiplet did not have an origin in the eleven dimen-

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sional supergravity theory. In particular, the brane charges have index structures that do not correspond to the coupling to any of the fields in the eleven dimensional supergravity theory.

The point particle, two and five branes of the eleven dimensional supergravity theory have charges that occur as central charges in the eleven dimensional supersymmetry algebra and indeed all such central charges have this interpretation. The dimensional reduction of these branes lead to branes whose charges also occur as central charges in the dimensional reduced supersymmetry algebra. Hence, the discovery mentioned above implies that the central charges of the dimensionally reduced supersymmetry algebra do not belong to E_8 Weyl multiplets. Although the supergravity theory is E_8 invariant, the gravitino, and so the spinorial supercharges transform, linearly only under the sub-algebra SO(16). Hence, from the perspective of the supergravity theory, there is no requirement for the central charges to belong to multiplets of E_8.

The above consideration apply just as well to dimensional reductions on tori of less than eight dimensions where the corresponding algebra is $E_d, d \leq 8$.

In this paper we adopt the viewpoint advocated in [20], namely that M theory is a non-linear realisation of E_8^{+++}. The fields of M theory belong to the adjoint representation of E_8^{+++}. At low levels these are precisely the fields of eleven dimension supergravity, including their duals, [20] and it is hoped that the fields at higher levels are dynamical and ensure the consistency of the theory. It was shown [26] that the Weyl transformations E_8^{+++} lead to transformations of the fields in the non-linear realisation that lead precisely to the transformations of equations (3.1) and (3.2). This is of course consistent with the work of references [15]–[18], but [26] provides a derivation of these formulae based on the non-linearly realised E_8^{+++} in contrast to the origins of these formulae in reference [15]–[18] which used the stringy property such as T-duality. In fact, the transformations of equation (3.1) are Weyl transformations corresponding to the simple roots on the gravity line and the transformations of equation (3.3) is just the Weyl transformation of the simple root of the exceptional node.

The fundamental representation of E_8^{+++} associated with its very extended node l_1, contains the space-time generators $P_{\hat{a}}, \hat{a} = 1, \ldots, 11$, which are the charges for the point particle of M theory, and the next two components in the l_1 multiplet contain the central charges of the eleven dimensional supersymmetry algebra which are the charges of the two and five brane respectively [22]. As such, we may conclude that this multiplet contains the "brane" charges of the full non-linearly realised E_8^{+++} theory. Indeed, as discussed in [23], it contains the correct A_{10} representations to be identified with the charges of all the solutions to the non-linear theory including those which are beyond those found in the eleven dimensional supergravity approximation. The decomposition of the l_1 representation into $E_8 \otimes A_2$ representations was studied in the previous section. It is the one appropriate for the dimensional reduction of the theory on a eight torus to three space-time dimensions and it allows us to read off the brane charges of the three dimensional theory in terms of multiplet of $E_8 \otimes A_2$. As discussed in appendix B, the latter factor, A_2, is related to the three dimensional Lorentz algebra. Since the l_1 representation has an infinite number of

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states, it will contain an infinite number of $E_8 \otimes A_2$ representations which are classified according to the level m_c. It is of course inevitable in this approach that the brane charges in three dimensions belong to multiplets of E_8.

While the previous section provides a systematic analysis of the decomposition we require, it is also instructive to consider the l_1 representations as it occurs in eleven dimensions and graded according to the level n_8, often called n_{11} in previous works, which is taken to be the number of time the root α_8 occurs. The states are then classified according to the representations of the algebra that remains once the node 8 is deleted, namely A_{10}. This was carried out for low levels in reference [22, 23] and, for convenience, we recall these low level charges

$$\begin{aligned}
&P_{\hat{a}}\ (0,2)\,,\ Z^{\hat{a}_1\hat{a}_2}\ (1,2)\,,\ Z^{\hat{a}_1\ldots\hat{a}_5}\ (2,2)\,,\ Z^{\hat{a}_1\ldots\hat{a}_7,b}\ (3,2)\,,\ Z^{\hat{a}_1\ldots\hat{a}_8}\ (3,0)\,,\\
&Z^{\hat{b}_1\hat{b}_2\hat{b}_3,\hat{a}_1\ldots\hat{a}_8}\ (4,2)\,,\ Z^{(\hat{c}\hat{d}),\hat{a}_1\ldots\hat{a}_9}\ (4,2)\,,\ Z^{\hat{c}\hat{d},\hat{a}_1\ldots\hat{a}_9}\ (4,0)\,,\ \ Z^{\hat{c},\hat{a}_1\ldots\hat{a}_{10}}\ (4,-2)\,,\ Z\ (4,-4)\\
&Z^{\hat{c},\hat{d}_1\ldots\hat{d}_4,\hat{a}_1\ldots\hat{a}_9}\ (5,2)\,,\ Z^{\hat{c}_1\ldots\hat{c}_6,\hat{a}_1\ldots\hat{a}_8}\ (5,2)\,,\ Z^{\hat{c}_1\ldots\hat{c}_5,\hat{a}_1\ldots\hat{a}_9}\ (5,0)\,,\\
&Z^{\hat{d}_1,\hat{c}_1\hat{c}_2\hat{c}_3,\hat{a}_1\ldots\hat{a}_{10}}\ (5,0)\,,\ Z^{\hat{c}_1\ldots\hat{c}_4,\hat{a}_1\ldots\hat{a}_{10}}\ (5,-2)\,,\ Z^{(\hat{c}_1\hat{c}_2,\hat{c}_3)}\ (5,2)\,,\ Z^{\hat{c},\hat{a}_1\hat{a}_2}\ (5,-2)\,,\\
&Z^{\hat{c}_1\ldots\hat{c}_3}\ (5,-4)\,.
\end{aligned} \tag{3.4}$$

In the above, the first figure in the brackets refers to the level n_8, while the second figure is the length squared β^2 of the root in E_8^{++++} to which the highest weight of A_{10} representation belongs. The index range is $\hat{a},\hat{b},\ldots = 1,\ldots,11$. The actual roots β can be found in reference [23]. We note that lower down in the list we find the state

$$Z^{\hat{d},\hat{c}_1\ldots\hat{c}_8,\hat{a}_1\ldots\hat{a}_8}\ (6,2) \tag{3.5}$$

Given the above list we may carry out the dimensional reduction to three dimensions "by hand" by dividing the index range $\hat{a} = (a,i)$ where $i,j,\ldots = 1,\ldots 8$ and $a,b,\ldots = 1,2,3$. The first set are A_7, or SL(8) indices, while the latter are the SL(3) indices. Although the eleven dimensional origin of the resulting states is clear from this method, the way the states package up into representations of E_8 is less clear. However, the above discussion will help us identify the eleven dimensional origin of the brane charges that we find in three space-time dimensions.

We now return to the $E_8 \otimes A_2$ decomposition of the l_1 representation in terms of the level m_c. For $m_c = 0$, we only have the solution discussed in the previous section, namely the only solution has $p_i = 0$ and $(q_1,q_2) = (1,0)$ with an E_8^{++++} root of $\beta = (1,0,\ldots,0)$ which has $\beta^2 = 2$. This is a singlet under E_8 and a vector under the three-dimensional Lorentz algebra; it is just the $P_a,\ a = 1,2,3$ in the first line of equation (3.4).

The solutions for $m_c = 1$ to equations (2.16), (2.17) and (2.19) for the case of $G = E_8$ are listed in table 4 given below.

The first column shows the $E_8 \otimes A_2$ representation content, the second column the E_8^{++++} root β for the highest weight state of $E_8 \otimes A_2$, the third column gives β^2 and the final column displays the the Sl(8) and SO(1,2) indices of the highest weight state.

The first multiplet in table 4 is a SO(1,2) singlet and so corresponds to charges of point particles in the three dimensional theory. They belong to the 248 representation of E_8 which

$E_8 \otimes A_2$	β	β^2	charge
$\lambda_1 \otimes (0,0)$	$\beta = (1,1,1,1,0,0,0,0,0,0,0,0)$	2	$Z^{i_1\ldots i_7}$
$1 \otimes (0,0)$	$\beta = (1,1,1,1,2,3,4,5,6,4,2,3)$	0	Z
$1 \otimes (1,1)$	$\beta = (1,0,0,1,2,3,4,5,6,4,2,3)$	2	$Z^a{}_b$

Table 4: Solutions for $m_c = 1$.

decomposes under SL(8) as $248 \to 8 + 28 + 56 + (63+1) + 56 + 28 + 8$. By examining the list of eleven dimensional charges of equation (3.4) and their E_8^{++++} roots given in reference [23] we can find how the point particle charges of the three dimensional arise in the eleven dimensional E_8^{+++} non-linearly realised theory. The highest weight component in table 4 has an E_8^{++++} root of $(1,1,1,1,0,0,0,0,0,0,0,0)$. and we see that this must be identified with the state P_4 in equation (3.4) which arises from the A_{10} highest weight state P_1 with root $(1,0^{11})$ by the action of $K^1{}_2, K^2{}_3$ and $K^3{}_4$. Hence, the first 8 of the 248 arise from the eleven dimensional theory as $P_a,\ a = 4,\ldots,11$. Similar considerations allow us to identify the 248 states and their E_8^{++++} roots in terms of the eleven dimensional brane charges of equation (3.4). Looking at the listing of roots on [23, page 22] we must look for roots that have $(1^4, n_1, n_2, \ldots, n_8)$ and are graded according to the construction of the root string of the adjoint representation of E_8. The relevant roots are easy to spot and are as follows;

$$\begin{aligned}
&P_i\ (8)(1^4,0^8)\,,\ Z^{ij}(28)(1^9,0^2,1)\,,\ Z^{i_1\ldots i_5}(56)(1^7,2,3,2,1,2)\,,\\
&Z^{i_1\ldots i_7,j}(63)(1^5,2,3,4,5,3,1,3)\,,\ Z^{i_1\ldots i_8}(1)(1^4,2,3,4,5,6,4,2,3)\,,\\
&Z^{i_1\ldots i_8,j_1\ldots j_3}(56)(1^4,2,3,4,5,6,4,2,4)\,,\\
&Z^{i_1\ldots i_6,j_1\ldots j_8}(28)(1^4,2,3,5,7,9,6,3,5)\,,\ Z^{i_1\ldots i_8,j_1\ldots j_8,k}(8)(1^4,3,5,7,9,11,7,3,6)
\end{aligned} \tag{3.6}$$

We note that the final 8 of the 248 occurs at a level of $n_8 = 6$ and so it is far above the branes whose charges occur in the eleven dimensional supersymmetry algebra. Indeed, the corresponding branes will couple to fields of the non-linear realisation which are well beyond those found in the usual supergravity approximation; for example, the branes associated with the final 8 couple to a field at level six in the non-linearly realised theory.

The second multiplet in table 4 is a singlet under both E_8 and SO(1,2). The final multiplet listed in table 4 has an E_8^{++++} root which does not have connected support on the E_8^{++++} Dynkin diagram and, as previously explained, it is not an acceptable root and so may be discarded.

The solutions for $m_c = 2$ to equations (2.16), (2.17) and (2.19) for the case of $G = E_8$ are listed in table 5.

The first multiplet in the above table is a first rank tensor of SL(3) and so a vector of the Lorentz algebra SO(1,2). It also transforms as the E_8 representation with highest weight state λ_7 which is the 3875 dimensional representation. When decomposed into A_8 representations the 3875 contains $8 + 70 + (216+8) + (28+36+420) + \ldots$. We note, from the above table, that it arises from an E_8^{++++} root that is related to the root $(1^9,0,0,1)$ for Z^{1011} of equation (3.4) by the addition of $(0^3,1^7,0,0)$ to the latter. As such, we find that

$E_8 \otimes A_2$	β	β^2	charge
$\lambda_7 \otimes (0,1)$	$\beta = (1,1,1,2,2,2,2,2,2,1,0,1)$	2	Z^{ai}
$\lambda_1 \otimes (0,1)$	$\beta = (1,1,1,2,2,3,4,5,6,4,3,2)$	0	$Z^{ai_1 \dots i_7}$
$\lambda_1 \otimes (2,0)$	$\beta = (1,0,1,2,2,3,4,5,6,4,3,2)$	2	$Z^{i_1 \dots i_7}_{(ab)}$
$1 \otimes (2,0)$	$\beta = (1,0,1,2,4,6,8,10,12,8,4,6)$	0	$Z_{(ab)}$
$1 \otimes (0,1)$	$\beta = (1,1,1,2,4,6,8,10,12,8,4,6)$	-2	Z^a
$1 \otimes (1,2)$	$\beta = (1,0,0,2,4,6,8,10,12,8,4,6)$	2	${Z_a}^{(bc)}$

Table 5: Solutions for $m_c = 2$.

this multiplet has a highest weight state that corresponds to Z^{311}. This is just part of Z^{ai} which are the first 8 of SL(8) in the decomposition of the 3875 representation. Carrying out the construction of the E_8 root string on λ_7 we find that the next highest A_8 state in the multiplet has an E_8^{++++} root of $(1^7, 2, 3, 2, 1, 2)$. On the other hand, the charge $Z^{7891011}$ of equation (3.4) has an E_8^{++++} root of $(1^7, 2, 3, 2, 1, 2)$. The difference between the roots is just that required to convert $Z^{7891011}$ to $Z^{3891011}$. This is part of the 70 states of $Z^{ai_1 \dots i_4}$ of the eleven dimensional theory which we identify with the next states in the λ_7 representation when decomposed into into A_8 multiplets. Proceeding in this way we can find all of the 3875 representation of E_8 in terms of A_8 representations and identify their origin in the eleven dimensional theory.

Thus we have found that the l_1 representation contains the point particle and string multiplets of brane charges found in references [15]–[18] deduced from the action of U-duality transformation on known brane charges. However, the l_1 representation contains an infinite number of E_8 multiplets and and so we can expect that all brane charges will be packaged together in this representation.

The solutions for $m_c = 3$ to equations (2.2.2), (2.2.3) and (2.2.5) for the case of $G = E_8$ are listed in table 3.3 given below.

We can use the equations in this paper to find the reduction to three dimensions of the non-linearly realised E_8^{+++} theory itself. In this case we just consider the decomposition of the adjoint representation of E_8^{+++} into $E_8 \otimes A_2$ representations and so instead of taking $m_* = 1$, as we did for the l_1 representations, we take $m_* = 0$. As noted in section 2, there are no solutions to the A_2 equation (2.23) for $m_c = 0, 1$, but there are two solutions for $m_c = 2$ which are given in table 2. We must combine these with the solutions of the E_8 equation (2.16) given in table 3. We then find that the solutions for $m_c = 2$ for the highest weights of $E_8 \otimes A_2$ in the decomposition of the adjoint representation of E_8^{+++} are given in the table 7 below.

Examining their E_8^{++++} roots we find that the last entry in the table arises from the eleven dimensional field A_{2311} and is the first component of the 3875 dimensional representation of E_8. The first and third entries, at level $n_8 = 3$, both arise from the eleven dimensional field $h^{a_1 \dots a_8, b}$ and in particular from the states $h^{35 \dots 11, 3}$ and $h^{35 \dots 11, 2}$ respectively. They are in the adjoint representation of E_8 whose first component is an 8 of SL(8) and these are given by $h^{i_1 \dots i_7 (a,b)}$ and $h^{i_1 \dots i_7 [a,b]}$ respectively. The first field is a

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$E_8 \otimes A_2$	β	β^2	charge
$\lambda_8 \otimes (1,0)$	$\beta = (1,1,2,3,3,3,3,3,3,2,1,1)$	2	Z_a
$\lambda_2 \otimes (0,2)$	$\beta = (1,1,1,3,3,3,4,5,6,4,2,3)$	2	$Z^{abi_1\ldots i_6}$
$2\lambda_1 \otimes (1,0)$	$\beta = (1,1,2,3,2,3,4,5,6,4,2,3)$	2	$Z_a^{i_1\ldots i_7,j_1\ldots j_7}$
$\lambda_2 \otimes (1,0)$	$\beta = (1,1,2,3,3,3,3,3,3,2,1,1)$	0	$Z_a^{i_1\ldots i_6}$
$\lambda_7 \otimes (2,1)$	$\beta = (1,0,1,3,4,5,6,7,8,5,2,4)$	2	${Z_{(ab)}}^{ci}$
$\lambda_7 \otimes (0,2)$	$\beta = (1,1,1,3,4,5,6,7,8,5,2,4)$	0	$Z^{(ab)i}$
$\lambda_7 \otimes (1,0)$	$\beta = (1,1,2,3,4,5,6,7,8,5,2,4)$	-2	Z_a^i
$1 \otimes (1,0)$	$\beta = (1,1,2,3,6,9,12,15,18,12,6,9)$	-6	Z_a
$1 \otimes (0,2)$	$\beta = (1,1,1,3,6,9,12,15,18,12,6,9)$	-4	$Z^{(ab)}$
$1 \otimes (2,1)$	$\beta = (1,0,1,3,6,9,12,15,18,12,6,9)$	-2	${Z^a}_{(bc)}$
$1 \otimes (1,3)$	$\beta = (1,0,0,3,6,9,12,15,18,12,6,9)$	2	${Z^{(abc)}}_d$
$\lambda_1 \otimes (1,0)$	$\beta = (1,1,2,3,4,6,8,10,12,8,4,6)$	-4	$Z_a^{i_1\ldots i_7}$
$\lambda_1 \otimes (0,2)$	$\beta = (1,1,1,3,4,6,8,10,12,8,4,6)$	-2	$Z^{(ab),i_1\ldots i_7}$
$\lambda_1 \otimes (2,1)$	$\beta = (1,0,1,3,4,6,8,10,12,8,4,6)$	0	$Z_{(ab)}^{c,i_1\ldots i_7}$

Table 6: Solutions for $m_c = 3$.

$E_8 \otimes A_2$	β	β^2	charge
$\lambda_1 \otimes (0,2)$	$\beta = (0,0,0,2,2,3,4,5,6,4,2,3)$	2	$F^{(ab)i_1\ldots i_7}$
$1 \otimes (0,2)$	$\beta = (0,0,0,2,4,6,8,10,12,8,4,6)$	0	$F^{(ab)}$
$\lambda_1 \otimes (1,0)$	$\beta = (0,0,1,2,2,3,4,5,6,4,2,3)$	0	$F_a^{i_1\ldots i_7}$
$1 \otimes (1,0)$	$\beta = (0,0,1,2,4,6,8,10,12,8,4,6)$	-2	F_a
$\lambda_7 \otimes (1,0)$	$\beta = (0,0,1,2,2,2,2,2,2,1,0,1)$	2	${F_a}^i$

Table 7: Solutions for $m_c = 2$.

symmetric second rank tensor under SL(3), but this becomes a reducible representation under the Lorentz algebra SO$(1,2)$. The trace part is just the usual adjoint representation of E_8 scalar fields one finds in the dimensional reduction to three dimensions of the eleven dimensional supergravity theory. A more detailed analysis of these fields, their dynamics and their relationship to the eleven dimensional theory will be given elsewhere.

4. Brane charges in nine dimensions and the relationships between the IIA and IIB theories

Another interesting dimension in which to compute the brane charges is nine space-time dimensions as this is the highest dimension in which the IIA and IIB supergravity theories dimensionally reduced on a torus coincide. We will first study this reduction from the point of view of the eleven dimensional theory, or equivalently, the IIA perspective. The decomposition relevant to the dimensional reduction to nine space-time dimensions on a torus corresponds to the deletion of the 9th node along the gravity line of the eleven dimensional theory in the equations of section 2 applied to the case of E_8^{+++}. The two algebras arising from the resulting Dynkin diagram are A_9 and A_1. The last node of A_9

is the exceptional node and hence only the first eight dots of the A_9 are on the gravity line of the nine dimensional theory and they correspond to the A_8 associated with space-time.

Adopting the notation of section 2 the roots and weights of A_9 are defined to be β_i and μ_i for $i = 1, 2, \ldots, 9$ respectively and for the A_1 are α and λ. For this case, $\alpha_c = x - \nu$ where $\nu = \mu_8 + \lambda$. One finds that $x^2 = -\frac{1}{10}$. Equation (2.13) become

$$\sum_i p_i (A^{A_9})^{-1}_{ij} - m_c (A^{A_9})^{-1}_{8j} - m_* (A^{A_9})^{-1}_{1j} = -m_j \,, \tag{4.1}$$

where p_i, m_j are positive integers and the fixed levels m_c and m_* are also positive integers. Equation (2.14) takes the simple form

$$p = -2m + m_c \,, \tag{4.2}$$

where p, m are positive integers and the level m_c is also a fixed positive integer. The latter equation is effectively already solved as one takes all values of p such that the right-hand side is positive.

The length squared of the E^{++++} root is given by equation (2.11) becomes

$$\beta^2 = \frac{3}{2} m_*^2 - \frac{(m_c + 2m_*)^2}{10} + \frac{p^2}{2} + \sum_{i,j=1}^{9} p_i ((A^{A_9})^{-1})_{ij} p_j = 2, 0, -2, \ldots \tag{4.3}$$

We are interested in the l_1 representation of E_{11} and so we take $m_* = 1$. The brane charges are classified in terms of $A_9 \otimes A_1$, however, only the A_8 sub-algebra of the A_9 is on the gravity line. We denote the generators of A_9 by $\hat{K}^a{}_b,\ a, b = 1, \ldots, 10$ where $a = 1, \ldots, 9$, are the indices corresponding to those of the nine dimensional space-time, and the final possible index value 10 refers to the exceptional node in the E_8^{+++} which belongs to the A_9 Dynkin sub-diagram and does not have any connection with the space-time index 10. The generators of A_9 are given, up to a factor, by

$$\hat{K}^a{}_b = K^a{}_b \,, \text{ for } a, b = 1, \ldots, 9 \text{ and } \hat{K}^a{}_{10} = R^{a1011} \,,\ \hat{K}^{10}{}_a = R_{a1011} \,, \text{ for } a = 1, \ldots, 9 \,, \tag{4.4}$$

where $K^a{}_b$, R^{a1011} and R_{a1011} are the usual generators of E_8^{+++} of the eleven dimensional theory. We place hats on the all objects associated with the representations of $A_9 \otimes A_1$. As explained in appendix B, the cartan involution invariant sub-algebra of A_8 is essentially the eight dimensional Lorentz algebra. The internal symmetry is just A_1.

For $m_c = 0$ we find only one solution of equation (4.1), namely $p_1 = 1$, all other Dynkin indices vanishing, and only one solution to equation (4.2), namely $p = 0$. This is an A_9 nine rank tensor, or a tensor with one lower index, but an A_1 singlet, $\hat{P}_a$. The corresponding E_8^{++++} root is $\beta = (1, 0^{11})$ which has length squared two. As such, we recognise $\hat{P}_a = P_a,\ a = 1, \ldots, 9$ and

$$\hat{P}_{10} = -[\hat{K}^a{}_{10}, P_{\tilde{a}}] = -[R^{a1011}, P_a] = -2Z^{1011} \tag{4.5}$$

we conclude that $P_{10} = 2Z^{1011}$ where Z^{1011} is the second member of the l_1 multiplet of the E_{11} viewed from the eleven dimensional perspective. It is just the two rank central charge

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of the eleven dimensional supersymmetry algebra. Here we have used the commutators of reference [22]. As such, for $m_c = 0$ the l_1 representation decomposed to $A_9 \otimes A_1$ contains the object $\hat{P}_a$ which consists of the eleven dimensional generators

$$P_a\,, \quad a = 1, \ldots, 9\,, \qquad \text{and} \qquad Z^{1011}\,. \tag{4.6}$$

For $m_c = 1$ we only find the solution of equation (4.1) namely $p_9 = 1$ all other Dynkin indices vanishing and only one solution to equation (4.2), namely $p = 1$. This is a A_9 one rank tensor and an A_1 doublet, $\hat{Z}^{ai}$, $a = 1, \ldots, 10$, $i = 1, 2$. The corresponding E_8^{++++} root is $\beta = (1^{10}, 0, 0)$ which has length squared two and corresponds to the highest weight state $\hat{Z}^{10\ 2}$. Using similar arguments to those deployed for the previous solution we find that $\hat{Z}^{ai}$, $a = 1, \ldots, 10$, $i = 1, 2$ contains, up to factors,

$$Z^{10\ 2} = P_{10}\,,\ Z^{10\ 1} = P_{11}\,,\ \hat{Z}^{a\ 2} = Z^{a\ 11}\,,\ \hat{Z}^{a\ 1} = Z^{a\ 10}\,, a = 1, \ldots, 9, i = 1, 2\,. \tag{4.7}$$

For $m_c = 2$, we find the solution of equation (4.1) which consists of $p_7 = 1$, all other Dynkin indices vanishing, together with the solution $p = 0$ to equation (4.2). This representation is an A_9 third rank tensor and an A_1 singlet, $\hat{Z}^{a_1a_2a_3}$, $a_1, a_2, a_3 = 1, \ldots, 10$. The corresponding E_8^{++++} root is $\beta = (1^8, 2, 2, 1, 1)$ which has length squared two and corresponds to the highest weight field $\hat{Z}^{91011}$. One finds, up to factors that $\hat{Z}^{a_1a_2a_3}$, $a_1, a_2, a_3 = 1, \ldots, 10$ consists of

$$\hat{Z}^{a_1a_210} = Z^{a_1a_2}\,,\ \hat{Z}^{a_1a_2a_3} = Z^{a_1a_2a_31011}\,, \quad a_1, a_2, a_3 = 1, \ldots, 9\,. \tag{4.8}$$

In fact, there are also another solutions with $m_c = 2$ to equations (4.1) and (4.2) but these does not satisfy equation (4.3). We have also is discarded other solutions to the above equations which do not have roots in E_8^{++++} which do not have a connected support on its Dynkin diagram. As discussed above, these do not correspond to actual roots of E_8^{++++}.

The above calculation computed the brane charges in nine dimensions of the eleven dimensional, or equivalently, the IIA theory in ten dimensions. However, we could also have considered the dimensional reduction to nine dimensions starting from the IIB non-linearly realised theory and reducing on a circle. The difference between the E_8^{+++} formulations that lead to the IIA and IIB theories is that the A_9 subalgebras associated with gravity are identified differently [20, 27, 32, 28]. For the IIA case, the A_9 gravity line contains the nodes labeled 1 to 9 along the horizontal line of the E_8^{+++} Dynkin diagram starting from the very extended node. While for the IIB theory, the A_9 gravity line of the E_8^{+++} Dynkin diagram contains the nodes labeled 1 to 8 which are along the horizontal line of the Dynkin diagram starting form the very extended node and the exceptional node labeled 11. To find the brane charges in the IIB theory in ten dimensions requires calculating the content of the l_1 representation in terms of the A_9 representation associated with the gravity line. This means we must consider the E_8^{++++} algebra with level one on the extra node, labeled * and then delete node ten. The brane charges are then classified in terms of the remaining D_{10} algebra. However, it is more convenient to consider a further decomposition by deleting node 9, whereupon the brane charges are classified by $A_9 \otimes A_1$ and labeled by the two levels m_9, m_{10}. However, this is in effect what we have just done in the calculation above, we deleted node 9 and the remaining algebra was $A_9 \otimes A_1$. The A_9 algebra is just that of the gravity line and so corresponds to the ten dimensional space-time in the IIB theory and the A_1 algebra has representations labeled by p, or equivalently m_{10}, but the latter is just the level associated with the node 10. Hence, we have already carried out the required decomposition of the l_1 in terms of A_9 with respect to the levels (m_9, m_{10}). The result was

$$\hat{P}_a\,,\ \hat{Z}^{ai}\,,\ \hat{Z}^{a_1a_2a_3}, \ldots \quad a=1,\ldots,10\,, \quad i=1,2\,. \tag{4.9}$$

We recognise that the l_1 representation of the IIB theory contains at low levels the space-time translations and the first two central charges that occur in the supersymmetry algebra of the IIB theory, Indeed, the hated notation was designed to be suitable for this interpretation of the calculation.

As explained in reference [28], the is a one to one relation between the three E_8^{+++} non-linear realisations associated with the eleven dimensional theory, the IIA theory and IIB theory which arises from their common E_8^{+++} origin. Indeed, the above identifications of the l_1 representations in the three theories given in equations(4.6)–(4.9), extend those given in this work. These identifications do not require any compactification of the three theories, but they also hold if the theories are compactified. In particular, we note that the the tenth component of the momentum of the IIB theory $\hat{P}^{10}$ is identified with the component Z^{1011} of the membrane charge, or equivalently, the charge of the string in the IIA theory. We recall that the momenta in the compactified dimensions are the charges associated with the Kaluza-Klein modes and that the topological charges associated with the winding modes of the string, or membrane, are the central charges in the compactified dimensions that occur in the supersymmetry algebra. In the nine dimensional theory, this means that the winding modes of the IIA string on the circle on which the IIA theory is reduced have a charge which is the Z^{1011} and so this must be identified with the Kaluza-Klein modes of the IIB theory whose charge is $\hat{P}_{10}$. Similarly, we also learn form equations (4.7) and (4.9) that the winding modes of the two stings in the IIB theory, whose charges are $\hat{Z}^{10i}$, must be identified with the Kaluza-Klein modes of the IIA theory reduced from eleven dimensions, whose charges are P_{10} and P_{11}.

Thus we recover the results of [39] which studied how the IIA and IIB supersymmetry algebras in ten dimensions lead to the unique nine dimensional supersymmetry algebra. As a result of these different origins, these authors were able to conclude that the Kaluza-Klein modes of the IIA string and those of the IIB string belonged to different supersymmetry multiplets and that T duality mapped the Kaluza-Klein modes of the IIA string into the string winding modes of the IIB string and that the winding modes of the IIA string were mapped to the Kaluza-Klein modes of the IIB theory. It is encouraging that these T-duality rules can be read off from their common E_8^{+++} origin in a simple way. Particularly, given that these results were derived in reference [39] using the string properties and supersymmetry algebra of these theories and these features have so far yet to be identified in the E_8^{+++} approach.

In reference [22] it was proposed to consider the non-linear realisation of the semi-direct product representation of E_8^{+++} and its l_1 representation. In this approach one has fields which are in a one to one correspondence with the generators of the Borel sub-algebra of

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E_8^{+++}, but also generalised coordinates which are in a one to one correspondence with the content of the l_1 representation. The fields then depend on these generalised coordinates. For the eleven dimensional theory, these generalised coordinates are $x^a, z_{ab}, z_{a_1\ldots a_5}, \ldots$. As explained in [28], when making the correspondence between the eleven dimensional, IIA and IIB theories one must not only swop the fields, but also exchange the generalised coordinates as explained above at low levels.

If we dimensionally reduce both the IIA and IIB non-linearly realised theories on a circle and keep all their dependence on the generalised coordinates, that is not only keep just the massless modes for example, we will find two theories in nine dimensions both of which have their original number of coordinates. In particular, the IIA theory will depend on the generalised coordinates x^a, $a = 1, \ldots, 9$, x^{10}, x^{11} , z_{1011} and an infinite number of other coordinates. While, the IIB theory in nine dimensions will have the generalised coordinates x^a, $a = 1, \ldots, 9$, x^{10}, $z_{10\ i}$ and an infinite number of other coordinates. Indeed, if we just consider the nine dimensional theory without worrying about keeping track of its higher dimensional origin then we must decompose he E_8^{++++} algebra with respect to the algebra that remains by deleting the nodes labeled eight, nine, ten and eleven with the corresponding level $(m_8, m_9, m_{10}, m_{11})$. We note that we delete the same node if we derive the theory from the IIA and IIB perspective and so the resulting theory derived from either path will be identical.

We have arrived at a theory which has some elements in common with that considered in references [40, 19]. These authors proposed to construct what was called a BPS extended nine dimensional theory consisting of nine dimensional supergravity coupled to the two towers of Kaluza-Klein multiplets as well a tower of states corresponding to the Kaluza-klein modes of the IIB string, or equivalently, the winding modes of the IIA string. Given the relations between these towers explained above, this nine dimensional theory is automatically T duality invariant. Given the charges associated with the towers it was proposed [19] that such a theory would arise by encoding three extra coordinates x^{10}, x^{11} and z_{1011}. However, this is just a restriction of the theory that would result from the dimensional reduction of the non-linear realisation of E_8^{+++}. However, this latter theory, unlike the BPS extended theory, would be ten dimensional Lorentz invariant. One may turn all this around and interpret this as evidence for the method of encoding space-time advocated in [22].

We close this section with a comment of the role of D_{10}. It is obvious that the E_8^{+++} non-linearly realised theory also possess a D_{10} symmetry as this algebra is obtained by deleting the node labeled 11 of the E_8^{+++} Dynkin diagram. This node is the last node in the gravity line associated with the eleven dimensional theory. As such, examining the D_{10} decomposition of the eleven dimensional theory by deleting this node destroys the manifest A_{10}, or SL(11), symmetry of the theory. Indeed, deleting this node leads to a residual gravity line with nine dots, or A_9, which is just the gravity line of the IIA theory. In fact, it is by deleting this node and examining the E_8^{+++} content in terms of the remaining algebra that gives the fields or generators IIA theory. Hence, it is natural to formulate the IIA theory in terms its D_{10} symmetry with the A_9 gravity line being part of this symmetry. Let us consider the l_1 representation from this perspective; its first components, P_a are the usual space-time translations and must be part of the fundamental representation of D_{10} associated with its first node, that is the one labeled one in the E_8^{+++} Dynkin diagram. However, this representation also contains a state Q^a which from the A_9 viewpoint has level one with respect to the one node not on the A_9 line of the D_{10} Dynkin diagram. In fact, it has an E_8^{++++} root of $(1^9, 0, 0, 1)$. This is just the central charge Z^a in the IIA supersymmetry algebra which arises from the eleven dimensional central charge Z^{a11}.

Deleting the node labeled 10 in the E_8^{+++} Dynkin diagram to find D_{10} also preserves a different A_9, denoted $\hat{A}_9$, which consists of the nodes labeled one to eight as well as node eleven. As explained above, this is just the $\hat{A}_9$ gravity line of the IIB theory and so this theory can also be formulated in terms of D_{10} which includes its A_9 gravity line. However, to deduce the $\hat{A}_9$ content of the theory we delete a different node of the E_8^{+++} Dynkin diagram as we did in the IIA case, namely the node labeled nine in the E_8^{+++} Dynkin diagram which is connected to the node we already delete to expose the D_{10} symmetry. Now the l_1 representation contains the usual space-time translations $\hat{P}_a$ of the IIB theory as well as a state $\hat{Q}^a$. However, in contrast to the IIA case, the highest weight state of $\hat{Q}^a$ has E_8^{++++} root $(1^{10}, 0, 0)$ and it is identified with the central charge $\hat{Z}^{a2}$ in the ten dimensional supersymmetry algebra.

5. Discussion and conclusion

In this paper we have derived equations which can be used to determine the decomposition of G^{+++} and its fundamental representation l_1, associated with the very extended node, into the sub-algebra whose Dynkin diagram is the one obtained from G^{+++} by deleting a node on the gravity line. These are the sub-algebras that arise when the non-linearly realised theory is dimensionally reduced. Following [23] we used the extended Dynkin diagram obtained by adding a further node to the very extended node by a single line to derive the content of the l_1 representation. Indeed, the level one, with respect to the new node, states in the adjoint representation of the extended algebra form the l_1 representation of G^{+++} . In fact, this technique can be used to discuss any representation of G^{+++}. If the representation has Dynkin indices p_j we just add a new node with p_j lines to the jth node of G^{+++} and consider the level one generators of this new algebra. Indeed, one may apply this method to any Kac-Moody algebra and not just very extended algebras.

We used the results of section 2 in section 3 to compute the contents of the l_1 representation in terms of $E_8 \otimes A_2$. This is the same as computing the E_8 multiplets of brane charges that occur when the non-linearly realised theory is reduced on an eight torus to three dimensions. At the lowest level, the point particle and string multiplets of charges we find are agreement with the previous results of references [15]–[18] which were derived by starting from some of the known charges and computing the remainder assuming that the U-duality transformations hold. The form of the U-duality transformations are just the T-duality transformations of string and the assumed $SL(2, Z)$ transformation of the IIB theory. However, as the authors of references [15]–[19] pointed out one finds in these multiplets many exotic charges which do not seem to arise from M theory as it has been previously discussed. However, from the view point of the eleven dimensional non-linearly realised E_8^{+++} theory their origin is clear. As observed in reference [23], there is a correspondence between the brane charges in the l_1 representation and the fields that appear in the non-linear realisation and hence for a given charge we know the field to which it couples. The mysterious charges can then be seen to couple to fields that are beyond those that occur in the supergravity approximation.

In the l_1 representation there are an infinite number of charges and so an infinite number of E_8 representations, however, it is encouraging to see how the different types of branes, i.e. point particles, strings, etc, all package up into this representation. the sympathetic reader can interpret these results as further evidence for the E_{11} symmetry underlying M theory.

In section 4, we performed a similar calculation for the nine dimensional theory and in particular traced how the brane charges arose from their different IIA and IIB origins. At low levels we find agreement with the correspondences found in references [19, 39, 40] using string theory and the supersymmetry algebra. Furthermore, adopting the approach to space-time advocated in reference [22] and making a considerable truncation we find a nine dimensional theory that makes contact with the type of BPS extended theory envisaged in [40, 19].

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A. A weights and inverse cartan matrix of E_n

The reader is invited to draw the Dynkin diagram of E_n. We draw $n-1$ dots connected by a horizontal line and then placing another dot (the exceptional node) above the third node from the right and connecting it with a single line to that node. We label the nodes in the horizontal line by $1, 2, \ldots, n-1$ from left to right and the node above the line by n. Following closely the techniques of reference [30], we use the decomposition of E_n to A_{n-1} given by deleting the exceptional node n. Let α_i and λ_i for $i = 1, \ldots, n-1$ be the simple roots and fundamental weights respectively of A_{n-1}. The roots of E_n can be written as [30]

$$\alpha_i\,, \qquad i = 1\,, \ldots, n-1, \qquad \alpha_n = x - \lambda_{n-3}\,, \tag{A.1}$$

where x is orthogonal to the roots of A_{n-1} and $x^2 = \frac{(9-n)}{n}$ in order that $\alpha_n^2 = 2$. The

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fundamental weights of E_n are given by

$$l_i = \begin{cases} \lambda_i + \frac{3i}{(9-n)}x, & i = 1, \ldots, n-3 \\ \lambda_i + \frac{(n-3)(n-i)}{(9-n)}x, & i = n-3, \ldots, n-1 \end{cases} \tag{A.2}$$

and

$$l_n = \frac{n}{(9-n)}x \,. \tag{A.3}$$

In deriving this result we used the scalar products of the fundamental weights of A_{n-1}

$$\lambda_i \cdot \lambda_j = (A^{A_{n-1}})^{-1}_{ij} = \frac{i(n-j)}{n} \,, \qquad \text{for} \qquad i \leq j \,. \tag{A.4}$$

The inverse cartan matrix of E_n is given by

$$\left(\left(A^{E_n} \right)^{-1} \right)_{ab} = l_a . l_b . \tag{A.5}$$

Using equation (A.4), we find the following algebraic formulae for the inverse cartan matrix of E_n

$$((A^{E_n})^{-1})_{ij} = \begin{cases} \frac{i(9-n+j)}{(9-n)}, & i,j = 1, \ldots, n-3, i \leq j \\ \frac{(n-j)((n-3)^2 - i(n-5))}{(9-n)}), & i,j = n-3, \ldots, n-1, i \leq j \\ 2\frac{i(n-j)}{(9-n)}, & i = 1, \ldots, n-3, j = n-3, \ldots, n-1, \end{cases} \tag{A.6}$$

and

$$((A^{E_n})^{-1})_{in} = \begin{cases} \frac{3i}{(9-n)}, & i = 1, \ldots, n-3 \\ \frac{(n-3)(n-i)}{(9-n)}, & i = n-3, \ldots, n-1 \end{cases} \tag{A.7}$$

and

$$((A^{E_n})^{-1})_{nn} = \frac{n}{(9-n)} \,. \tag{A.8}$$

It is easy to check that for $n = 8, 10$ the inverse cartan matrix has integer valued entries, if $n \leq 8$ they are positive and for $n = 10$ they are negative.

B. Cartan involution invariant sub-algebra of A_n and G^+

Given a Kac-Moody algebra, the cartan involution is defined to act on the Chevalley generators E_a, F_a and H_a as

$$E_a \to -F_a, \; F_a \to -E_a, H_a \to -H_a \,. \tag{B.1}$$

As such, the sub-algebra invariant under the cartan involution is generated by

$$E_a - F_a \,. \tag{B.2}$$

The generators of A_n are $K^a{}_b,\ a,b = 1,2,\ldots,n+1$ and obey the relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,. \tag{B.3}$$

The Chevalley generators are

$$E_a = K^a{}_{a+1},\ F_a = K^{a+1}{}_a,\ H_a = K^a{}_a - K^{a+1}{}_{a+1}\,. \tag{B.4}$$

The reader may readily verify that they do satisfy the defining relations of the Kac-Moody algebra corresponding to the cartan matrix of A_n.

The cartan involution defined above induces an action on all the generators of A_n as follows;

$$K^a{}_b \to -K^b{}_a \tag{B.5}$$

and as such the cartan involution invariant sub-algebra has the generators

$$J_{ab} = K^a{}_b - K^b{}_a\,. \tag{B.6}$$

It is straightforward to verify using equation (B.3) that J_{ab} obey the commutation relations of $\mathrm{SO}(n+1)$. As such, we recover the known fact that the cartan involution invariant sub-algebra of $\mathrm{SL}(n+1)$ is $\mathrm{SO}(n+1)$.

A generator in the vector representation of A_n, i.e. with non-vanishing Dynkin index $p_n = 1$, transforms under A_n as

$$[K^a{}_b, R^c] = \delta^c_b R^a - \frac{\delta^a_b}{n+1} R^c\,, \tag{B.7}$$

with $K^a{}_b$ acting in a similar way for more complicated tensors. It is easy to verify that

$$[J_{ab}, R^c] = \delta^c_b R^a - \delta^c_a R^b \tag{B.8}$$

with a similar action on more complicated tensors. Hence, a tensor under $\mathrm{SL}(n+1)$ transforms under its cartan involution invariant sub-algebra, $\mathrm{SO}(n+1)$ as the indices suggest. However, as δ_{ab} is an $\mathrm{SO}(n+1)$ invariant tensor, the representation of $\mathrm{SL}(n+1)$ is not always an irreducible representation of $\mathrm{SO}(n+1)$. For example, the tensor $T^{(ab)}$ of $\mathrm{SL}(n+1)$ with non-vanishing Dynkin index $p_n = 2$ transforms irreducibly under $\mathrm{SO}(n+1)$ i.e. as singlet and a symmetric traceless tensor, $T^{(ab)} - \frac{\delta^{ab}}{n+1} T^c{}_c$.

By introducing a suitable number of minus signs into the cartan involution of equation (B.1) and repeating the above steps one finds that the invariant sub-algebra is $\mathrm{SO}(p, n+1-p)$. Rather than continually record these minus signs, it is more efficient to just remember that one should put them in, but to stick to the signs of equation (B.1) and simply then restore the final result to what it should be by simply carry out a Wick rotation on the final result.

The above discussion plays an important role in this paper as the space-time Lorentz algebra arises in the non-linear realisation just as the above suggests, namely as the cartan involution invariant sub-algebra of the A_D associated with the gravity line. As such, the Lorentz algebra in the dimensionally reduced theory is just the cartan involution invariant sub-algebra of the A_D associated with the gravity line which remains to the left of the node that is deleted. The representations of the Lorentz group are then read off from those of SL$(n+1)$ as described above.

In the remainder of this appendix we give the cartan involution invariant sub-algebra of G^+. This is work carried out with Matthias Gaberdiel and a more detail account will be given elsewhere [41]. Let us consider any finite dimensional semi-simple Lie algebra G with generators E_α and H_a where α is any root. Under the cartan involution of equation (B.1) these transform as $E_\alpha \to -E_{-\alpha}$ and $H_a \to -H_a$. The roots of the affine algebra G^+ can be written in the form $(\alpha, 0, n)$ and $(0, 0, n)$ and under the cartan involution they transform as to $(-\alpha, 0, -n)$ and $(0, 0, -n)$ respectively. The corresponding generators are $E_{\alpha,n}$ and $H_{a,n}$ and, together with the central generator k, they transform as

$$E_{\alpha,n} \to -E_{-\alpha,-n}\,, \quad H_{a,n} \to -H_{a,-n} \qquad \text{and} \qquad k \to -k\,. \tag{B.9}$$

Taking the loop approach to the affine algebra we may write the generators in the form

$$E_\alpha(x) = \sum_n E_{\alpha,n} x^{-n}, \; H_a(x) = \sum_n H_{a,n} x^{-n}\,, \tag{B.10}$$

where $x = e^{i\theta}$ for $-\pi \le \theta \le \pi$. Under the cartan involution

$$E_\alpha(x) \to -E_{-\alpha}(x^{-1})\,, \qquad H_a(x) \to -H_a(x^{-1})\,. \tag{B.11}$$

Hence, the cartan involution invariant sub-algebra contains the combinations

$$K_\alpha(x) = E_\alpha(x) - E_{-\alpha}(x^{-1})\,, \qquad \text{and} \qquad L_a(x) = H_a(x) - H_a(x^{-1})\,. \tag{B.12}$$

We note that the central generator has been eliminated. It is more useful to work with combinations that have a definite symmetry under $\theta \to -\theta$ and so we consider the cartan involution invariant sub-algebra to contain

$$Q_\alpha = K_\alpha(x) - K_\alpha(x^{-1})\,, \qquad P_\alpha = K_\alpha(x) + K_\alpha(x^{-1})\,, \qquad L_a(x)\,. \tag{B.13}$$

We can construct the cartan involution invariant sub-algebra of G^+ directly from G by considering an interval algebra rather than the loop algebra which leads to G^+. We divide the generators of G into those that are eigenstates of the cartan involution, namely $P_\alpha = E_\alpha - E_{-\alpha}$, $Q_\alpha = E_\alpha + E_{-\alpha}$ and H_a. We then consider the mapping from the interval $[0, \pi]$ into the group G to definite the generators $P_\alpha(\theta),\ Q_\alpha(\theta),\ H_a(\theta)$ which are subject to the boundary conditions

$$\frac{dP_\alpha(\theta)}{d\theta} = 0, Q_\alpha(\theta) = 0\,, \qquad H_a(\theta) = 0 \tag{B.14}$$

at $\theta = 0, \pi$. Hence, $P_\alpha(\theta)$ obeys Neumann boundary conditions while $Q_\alpha(\theta)$ and $H_a(\theta)$ obey Dirichlet boundary conditions. We can extend the range of the interval to be from $-\pi$ to π by demanding that $P_\alpha(\theta) = P_\alpha(-\theta)$, $Q_\alpha(\theta) = -Q_\alpha(-\theta)$, $H_a(\theta) = -H_a(-\theta)$.

In fact, the cartan involution invariant sub-algebra of G^+ is not a Kac-Moody algebra as can be shown by finding the invariant bilinear form on the algebra and showing that it is not non-degenerate.

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